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Ozen, U.; Norde, H.W.; Slikker, M.

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By Ulaş Özen, Henk Norde, Marco Slikker

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On the convexity of newsvendor games¹

Ulaş Özen • Henk Norde • Marco Slikker

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Department of Technology Management, Technische Universiteit Eindhoven, P.O. Box 513, NL-5600
MB Eindhoven, Netherlands

Department of Econometrics and Operations Research, Tilburg University, P.O. Box 90153, NL-5000
LE Tilburg, Netherlands

Department of Technology Management, Technische Universiteit Eindhoven, P.O. Box 513, NL-5600
MB Eindhoven, Netherlands

U.Ozen@tm.tue.nl • h.norde@uvt.nl • M.Slikker@tm.tue.nl

Abstract

This study considers a simple newsvendor situation that consists of n retailers, all selling the same item with common purchasing costs and common selling prices. Groups of retailers might increase their expected joint profit by inventory centralization, which means that they make a joint order to satisfy total future demand. The resulting newsvendor games are shown to have non-empty cores in the literature. This study investigates convexity of newsvendor games. We focus our analysis on the class of newsvendor games with independent symmetric unimodal demand distributions after providing several examples outside this class that are not convex. Several interesting subclasses, containing convex games only, are identified. Additionally, we illustrate that these results can not be extended to all games in this class.

Keywords: inventory centralization, game theory, newsvendor and convexity.

JEL classification code: C71.

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1 Introduction

It has been well studied in the inventory literature that inventory centralization leads to cost reduction or profit increase in an environment with multiple newsvendors (see, e.g., Eppen (1979), and Chen and Lin (1989)). Afterwards, several papers have analyzed the problem of allocation of the gains from inventory centralization using the tools provided by cooperative game theory, mainly focusing on the core. Hartman et al. (2000) studied multiple newsvendor situations and showed that associated newsvendor games have non-empty cores under specific assumptions on demand distributions. Then Müller et al. (2002) came with a more powerful result proving that newsvendor games have non-empty cores regardless of the demand distributions. The same result was independently derived by Slikker et al. (2001).

Several generalizations of the simple multiple newsvendor model considered above have been studied in the literature as well. Slikker et al. (2005) considered a model with nonidentical wholesale and customer prices for the newsvendors. Moreover, they incorporate transshipment costs in their model. They showed that the associated newsvendor games with transshipments have non-empty cores. Afterwards, Özen et al. (2004) showed a similar result for a multiple newsvendor situation with warehouses.

Besides the core concept, cooperative game theory provides other solution concepts for the benefit allocation problem as well. These solution concepts haven been paid attention to by several works dealing with the allocation of the gains from inventory centralization in the literature. Gerchak and Gupta (1991) considered inventory centralization in the context of a continuous review policy inventory system with complete back-ordering. After showing that inventory centralization always results in lower cost, they compared four simple allocation mechanisms and showed that only one of them guarantees lower cost for every store than its stand-alone cost. Robinson (1993) extended their analysis in terms of the core to other allocation mechanisms, i.e., the Shapley value (cf. Shapley (1953)) and the Lounderback allocation (Lounderback (1976)). Hartman and Dror (1996) examined allocation mechanisms for this setting using three criteria. These are core non-emptiness, computational ease and justifiability, which implies that the allocation of cost in the core of the cost game corresponds to the allocation of benefits in the core of the benefit game. Finally, we mention two papers utilizing a hybrid analysis, Anupindi et al. (2001), and Granot and Sosic (2003). They studied a model, where different retailers make their ordering decisions independently\non-cooperatively and transshipment of these orders

take place cooperatively. They mainly focus on the question of what kind of allocation mechanisms in the cooperative transshipment game might lead to joint optimal order quantities being an equilibrium.

In this work, we study convexity of simple newsvendor games. Convex games are well-known for having several nice properties related to solution concepts. First of all, Shapley (1971) and Ichiischi (1981) showed that the marginal vectors of a game are the extreme points of the core if and only if the game is convex. Besides, the bargaining set coincides with the core. With respect to one-point solution concepts, it holds that the Shapley value is the barycenter of the core. Furthermore, the kernel coincides with the nucleolus (Maschler et al.(1972)) and the τ -value can easily be calculated (Tijs (1981)). This paper fits in the literature of operations research games (*OR games*), which are cooperative games arising from operations research problems. See Borm et al. (2001) for a survey on OR games. Convexity of OR games have been paid special interest by several authors. We name a few of them here. Hamers et al. (2005) and Borm et al. (2002) studied convexity of games corresponding to different sequencing situations and Granot et al. (2002) showed that extended tree games are convex which helps them to derive algorithms to compute the Shapley value and nucleolus of extended tree games.

It is known in the literature that newsvendor games are not convex in general (see Hartman and Dror (1997) and Slikker et al. (2001)). In this study, we focus on the class of newsvendor games with independent symmetric unimodal demand distributions. Several subclasses are shown to contain convex games only. Surprisingly, however, these results cannot be generalized to the whole class. A counterexample is provided.

The organization of this paper is as follows. In section 2, we give preliminaries on cooperative game theory and introduce some notation. Furthermore, we describe newsvendor games here to make the paper self-contained. In section 3, we focus on newsvendor situations with independent symmetric unimodal demand distributions and we show that their associated games are convex if the optimal fractile equals $1/2$. In section 4, we focus on newsvendor situations with normal demand distributions and we prove that the associated games are convex. In section 5, we investigate the marginal contribution of retailers with uniform demand distributions to small and big coalitions. Then in section 6, we show by means of a counterexample that newsvendor games with independent symmetric unimodal demand distributions are not necessarily convex. We conclude the paper with some remarks in section 7.

2 Preliminaries

In this section, we recall some notions from cooperative game theory and introduce newsvendor games, some definitions and notations.

Let N be a finite set of players, $N = \{1, \dots, n\}$. A subset of N is called a *coalition* and denoted by S . A function v , assigning a value $v(S)$ to every coalition $S \subseteq N$ with $v(\emptyset) = 0$, is called a *characteristic function*. The value $v(S)$ is interpreted as the maximum total profit that coalition S can obtain through cooperation. Assuming that the benefit of a coalition S can be transferred between the players of S , a pair (N, v) is called a *cooperative game with transferable utility* (TU game). The *core* of a game (N, v) is the set

$$\text{Core}(v) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for every } S \subseteq N\}.$$

Intuitively, the core of a game can be interpreted as the set of payoff vectors for which no coalition has an incentive to leave the grand coalition N . Note that the core of a game can be empty.

Two interesting properties that a game might satisfy are superadditivity and convexity. A game (or its characteristic function) is called *superadditive* if for any two disjoint coalitions S and T it holds that $v(S) + v(T) \leq v(S \cup T)$. For superadditive games, it is always attractive for two disjoint coalitions to form one big coalition. We remark that superadditive games do not necessarily have non-empty cores. A game (N, v) is called *convex* if

$$\begin{aligned} v(T \cup \{i\}) - v(T) &\geq v(S \cup \{i\}) - v(S) \\ \text{for all } i \in N \text{ and all } S, T \subseteq N \setminus \{i\} \text{ with } S \subset T. \end{aligned} \tag{1}$$

Hence, for convex games, the marginal contribution of a player to any coalition is greater than his marginal contribution to a smaller coalition. A game is *strictly convex* if all inequalities are strict. We remark that convex games have non-empty cores.

Before introducing newsvendor games, we first introduce newsvendor situations. A *newsvendor situation* is defined as a tuple $(N, (X_i)_{i \in N}, c, p)$, where

- N : Set of retailers, $N := \{1, \dots, n\}$;
- X_i : Stochastic demand at retailer i , with $E[X_i] < \infty$ for every $i \in N$;
- c : Common transportation cost of goods from supplier to the retailers;
- p : Common selling price of the goods at the retailers.

Throughout the study, we assume that p and c are positive and $p \geq c$. Let F_i be the distribution function of X_i ². Consider a retailer $i \in N$. Note that, being stand-alone, this retailer solves a standard newsvendor problem with demand X_i , selling price p and purchasing cost c while determining its order quantity. Consider a collection of retailers $S \subseteq N$, who form a coalition and jointly determine an order quantity to satisfy their joint demand. This stochastic demand is given by $X_S = \sum_{i \in S} X_i$. Furthermore, F_S denotes the distribution function of X_S .

Let x_S be a realization of random demand X_S and let $q \in \mathbf{R}$ be a joint order quantity. The profit of coalition S for x_S and q is given by

$$r^S(q, x_S) = -cq + p \min\{q, x_S\} \quad (2)$$

and the expected profit function of coalition is defined by

$$\pi^S(q) = E_{X_S}[r^S(q, X_S)].$$

The associated newsvendor game (N, v) is defined as follows:

$$v(S) = \max_q \pi^S(q) \quad \text{for all } S \subseteq N. \quad (3)$$

In other words, the characteristic function v assigns to a coalition the maximum expected profit this coalition can obtain.

We remark that the expected profit function π^S is a newsvendor type profit function and coalition S solves a standard newsvendor problem with demand X_S , selling price p and purchasing cost c . Suppose that X_i for all $i \in N$ are continuous and let f_i and f_S denote the density functions of X_i and X_S , respectively. Then, the optimal order quantity q^S of coalition S , which maximizes his expected profit function, is the one satisfying the well-known fractile equality $F_S(q^S) = 1 - c/p$ (see Silver et al. (1998) for determination of optimal order quantity, when demand distributions are discrete and see Khouja (1999) for a literature review on newsvendor models)³. We call $1 - c/p$ the optimal fractile.

²In most practical applications X_i can not take negative values. However, in this work we allow X_i to take negative values with very low probabilities to cover well known distributions (e.g., normal distribution). Besides, negative demand can be interpreted as returns from customers.

³Since we assume that demand can take negative values, optimal order quantities can take negative values too. In these situations, the profit function (2) is unrealistic, i.e., the retailer sells the amount of negative order of imaginary goods to the supplier. We include these situations in the analysis for technical reasons. But our primary interest is in the situations, where optimal order quantities are positive.

In this case, we can write that

$$\begin{aligned}
v(S) &= -c * q^S + pE_{X_S}[\min(q^S, X_S)] \\
&= -c * q^S + p \int_{-\infty}^{q^S} x f_S(x) dx + p \int_{q^S}^{\infty} q^S f_S(x) dx \\
&= -c * q^S + p \left(q^S F_S(q^S) - \int_{-\infty}^{q^S} F_S(x) dx \right) + pq^S(1 - F_S(q^S)) \\
&= -c * q^S - p \int_{-\infty}^{q^S} F_S(x) dx + pq^S \\
&= -c * q^S - p \left(q^S(1 - \frac{c}{p}) - \int_0^{F_S(q^S)} F_S^{-1}(y) dy \right) + pq^S \\
&= p \int_0^{1 - \frac{c}{p}} F_S^{-1}(y) dy \quad \text{for all } S \subseteq N.
\end{aligned} \tag{4}$$

The first equality follows since q^S is the optimal order quantity. The third equality holds by means of partial integration. The fifth equality follows from interchanging the axes of integration⁴. The last equality holds by $q^S = F_S^{-1}(1 - c/p)$. Note that $v(S)$ is linearly dependent on the pair (c, p) , i.e., if both c and p change by a factor λ , the optimal fractile remains unchanged and $v(S)$ is multiplied by λ as well.

Consider a group of retailers N facing random demand $(X_i)_{i \in N}$ with a selling price p . We define the following general characteristic function, which assigns a value to each coalition $S \subset N$ for each optimal fractile $y \in [0, 1]$, as follows:

$$v_S(y) = p \int_0^y F_S^{-1}(z) dz. \tag{6}$$

Note that for the newsvendor game (N, v) associated with newsvendor situation $(N, (X_i)_{i \in N}, c, p)$, it holds that $v_S(1 - c/p) = v(S)$.

In the literature, it is shown that newsvendor games have non-empty cores in general (Müller et al. (2002) and Slikker et al. (2001)). Moreover, newsvendor games are superadditive.

Consider a coalition S with demands X_i , $i \in S$, that are independently distributed. Then the distribution of $X_S = \sum_{i \in S} X_i$ can be determined by the well-known convolution

⁴We remark that F_S is a continuous and weakly increasing function. Hence, its inverse might be multi-valued. Choosing any of these values as a principal value yields the same result.

formula:

$$F_S(x) = \int_{-\infty}^{\infty} f_Z(u) F_{S \setminus Z}(x - u) du, \quad (7)$$

for any non-empty $Z \subset S$.

In this paper, we analyze the convexity of newsvendor games. We especially focus on newsvendor situations with symmetric and unimodal demand distributions. We use the following definitions for unimodality and symmetry.

Definition 1 *A continuous random variable X is unimodal if there exists $a \in \mathbb{R}$ such that its distribution function F_X is convex on $(-\infty, a]$ and concave on $[a, \infty)$. A continuous stochastic variable X is called strictly unimodal if it is unimodal and its density function f_X has a unique maximum at a . The distribution function F_X and the density function f_X are called (strictly) unimodal as well.*

Definition 2 *A continuous random variable X is symmetric if there exists $a \in \mathbb{R}$ such that $f_X(a - x) = f_X(a + x)$ for all $x \geq 0$, i.e., $F_X(a - x) + F_X(a + x) = 1$ for all $x \geq 0$. Note that $a = \mu$ with $\mu = E[X]$. The distribution function F_X and the density function f_X are called symmetric as well.*

Definition 3 *A discrete random variable X is symmetric if there exists $a \in \mathbb{R}$ such that for its distribution function F_X , $F_X(a - x) + F_X(a + x) = 1$ for all $x \geq 0$. Note that $a = \mu$ with $\mu = E[X]$.*

From Wintner (1938), it follows that the sum of continuous random variables with independent symmetric unimodal distributions is unimodal. Moreover, by the convolution formula, it can be easily shown that this sum is symmetric as well.

Finally, we introduce some notation. A normal distribution with mean μ and standard deviation σ is denoted by $Norm(\mu, \sigma)$. Θ and θ denote the distribution function and density function of the standard normal distribution $Norm(0, 1)$, respectively. A uniform distribution on range $[a, b]$ is denoted by $U(a, b)$. Superscript (k) of a function denotes the k^{th} derivative of the function.

3 Symmetric unimodal demand distributions

In this section, we first present some examples that show that newsvendor games are not convex in general. Afterwards, we focus on newsvendor situations with independent

symmetric unimodal demand distributions and we show that their associated games are convex if the optimal fractile equals $1/2$. Finally, we present a proposition stating that the convexity of newsvendor games with symmetric demand distributions are symmetric around fractile $1/2$, i.e., for two newsvendor situations $\Gamma = (N, (X_i)_{i \in N}, c, p)$ and $\bar{\Gamma} = (N, (X_i)_{i \in N}, \bar{c}, p)$ with $\bar{c} = p - c$, the associated games are either both convex or not.

Though newsvendor games have non-empty cores, they do not need to be convex even for very simple settings. The following example considers such a setting with identical retailers having independent symmetric discrete demand distributions.

Example 3.1 Consider newsvendor situation $\Gamma = (N, (X_i)_{i \in N}, c, p)$, such that $N = \{1, 2, 3\}$, independent stochastic demands X_i with probability mass function

$$p_i(x) = \begin{cases} 0 & \text{if } x \notin \{1, 2\}; \\ \frac{1}{2} & \text{if } x = 1; \\ \frac{1}{2} & \text{if } x = 2, \end{cases}$$

for all $i \in N$, $c = 1$ and $p = 2$. By (3), it is a straightforward exercise to determine the associated newsvendor game, which is described by

$$v^\Gamma(S) = \begin{cases} 1 & \text{if } |S| = 1; \\ 2.5 & \text{if } |S| = 2; \\ 3.75 & \text{if } S = N. \end{cases}$$

Hence, with $i = 3$, $S = \{1\}$ and $T = \{1, 2\}$ we derive that

$$v^\Gamma(S \cup \{i\}) - v^\Gamma(S) = 1.5 > v^\Gamma(T \cup \{i\}) - v^\Gamma(T) = 1.25.$$

Therefore, we conclude that (N, v^Γ) is not convex.

◇

The following example shows that newsvendor games with dependent symmetric unimodal demand distributions are not convex in general.

Example 3.2 Consider newsvendor situation $\Gamma = (N, (X_i)_{i \in N}, c, p)$, such that $N = \{1, 2, 3\}$, $c = 1$, and $p = 2$. Let X_1 be uniformly distributed on $[0, 1]$, and let $X_2 = X_1$ and $X_3 = 1 - X_1$. In other words, X_1 and X_2 are positively correlated, X_1 and X_3 are negatively correlated, and X_2 and X_3 are negatively correlated.

It is a straightforward exercise to determine the associated newsvendor game, which is described by

$$v^\Gamma(S) = \begin{cases} 0.25 & \text{if } |S| = 1; \\ 0.5 & \text{if } S = \{1, 2\}; \\ 1 & \text{if } S = \{1, 3\} \text{ or } S = \{2, 3\}; \\ 1.25 & \text{if } S = N. \end{cases}$$

Hence, with $i = 1$, $S = \{3\}$ and $T = \{2, 3\}$ we derive that

$$v^\Gamma(S \cup \{i\}) - v^\Gamma(S) = 0.75 > 0.25 = v^\Gamma(T \cup \{i\}) - v^\Gamma(T).$$

Therefore, we conclude that (N, v^Γ) is not convex.

◇

This example can easily be adjusted to cover dependent normal demand distributions.

Before focusing on newsvendor games with independent symmetric unimodal demands, we present a lemma that deals with the change in profit as a result of forming a bigger coalition.

Consider two coalitions S and T such that $S, T \subset N$ and $S \cap T = \emptyset$. Let q^S , q^T and $q^{S \cup T}$ denote the optimal order quantities of coalitions S , T and $S \cup T$, respectively. Let $G(S, T)$ be the expected extra benefit if coalitions S and T come together and form coalition $S \cup T$, i.e., $G(S, T) = v(S \cup T) - v(S) - v(T)$. The following lemma shows that the change in profit as a result of forming a bigger coalition consists of the change in cost because of ordering optimally for coalition $S \cup T$, the expected extra revenue coming from the exchange of excess goods to satisfy excess demand with an order size of the sum of optimal orders of the contributing coalitions and the expected extra revenue of ordering optimally instead of the sum of optimal orders of the contributing coalitions.

Lemma 1 *Consider a newsvendor situation $(N, (X_i)_{i \in N}, c, p)$. Let $S, T \subseteq N$ such that $S \cap T = \emptyset$. Then*

$$\begin{aligned} G(S, T) &= -c * (q^{S \cup T} - q^S - q^T) + p * E_{X_{S \cup T}}[(X_{S \cup T} - (q^S + q^T))^+ - (X_{S \cup T} - q^{S \cup T})^+] \\ &\quad + p * E_{X_{S \cup T}}[\min((q^S - X_S)^+, (X_T - q^T)^+)] \\ &\quad + p * E_{X_{S \cup T}}[\min((q^T - X_T)^+, (X_S - q^S)^+)]. \end{aligned}$$

The proof of this lemma is given in the appendix.

Our first theorem states that newsvendor games with independent symmetric unimodal demand distributions are convex if optimal fractile $1 - c/p$ equals $1/2$.

Theorem 1 *Let $(N, (X_i)_{i \in N}, c, p)$ be a news-vendor situation. If the random demands $(X_i)_{i \in N}$ have independent, symmetric and unimodal distributions, and additionally $p = 2 * c$, then the associated newsvendor game is convex.*

Proof: To prove this theorem, we will show that

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T) \text{ for all } S, T \subset N, i \in N \text{ such that } S \subset T \subseteq N/\{i\}.$$

Consider a coalition $S \subseteq N$. Since the demand distributions of the retailers in coalition S are independent and symmetric, the demand distribution of coalition S is also symmetric. Using $p = 2 * c$ as well, we find that the optimal order quantity satisfies $q^S = F_S^{-1}(1/2) = \mu_S = E[X_S] = \sum_{j \in S} E[X_j] = \sum_{j \in S} \mu_j$. Without loss of generality, we assume that $p = 1$ in the remaining part of the proof.

Let $i \in N$ and $S \subseteq N/\{i\}$. Then

$$\begin{aligned} G(S, i) &= -\frac{1}{2} * (\mu_{S \cup \{i\}} - \mu_S - \mu_i) + E_{X_{S \cup \{i\}}}[(X_{S \cup \{i\}} - (\mu_S + \mu_i))^+ - (X_{S \cup \{i\}} - \mu_{S \cup \{i\}})^+] \\ &\quad + E_{X_{S \cup \{i\}}}[\min((\mu_i - X_i)^+, (X_S - \mu_S)^+)] \\ &\quad + E_{X_{S \cup \{i\}}}[\min((\mu_S - X_S)^+, (X_i - \mu_i)^+)] \\ &= E_{X_{S \cup \{i\}}}[\min((\mu_i - X_i)^+, (X_S - \mu_S)^+)] \\ &\quad + E_{X_{S \cup \{i\}}}[\min((\mu_S - X_S)^+, (X_i - \mu_i)^+)] \\ &= \int_{-\infty}^{\mu_i} \left(\int_{\mu_S}^{\mu_S + \mu_i - x} (z - \mu_S) f_S(z) dz + (\mu_i - x) * (1 - F_S(\mu_S - x + \mu_i)) \right) f_i(x) dx \\ &\quad + \int_{\mu_i}^{\infty} \left(\int_{\mu_S + \mu_i - x}^{\mu_S} (\mu_S - z) f_S(z) dz + (x - \mu_i) * F_S(\mu_S - (x - \mu_i)) \right) f_i(x) dx \\ &= 2 \int_{-\infty}^{\mu_i} \left(\int_{\mu_S}^{\mu_S + \mu_i - x} (z - \mu_S) f_S(z) dz + (\mu_i - x) * (1 - F_S(\mu_S - x + \mu_i)) \right) f_i(x) dx \\ &= 2 \int_{-\infty}^{\mu_i} \left(\int_{\mu_S}^{\mu_S + \mu_i - x} (z - \mu_S) f_S(z) dz + (\mu_i - x) * \int_{\mu_S + \mu_i - x}^{\infty} f_S(z) dz \right) f_i(x) dx \\ &= 2 \int_0^{\infty} \left(\int_0^Q y f_S(y + \mu_S) dy + \int_Q^{\infty} Q f_S(y + \mu_S) dy \right) f_i(\mu_i - Q) dQ \\ &= 2 \int_0^{\infty} \left(\int_0^Q \int_0^y f_S(y + \mu_S) dx dy + \int_Q^{\infty} \int_0^Q f_S(y + \mu_S) dx dy \right) f_i(\mu_i - Q) dQ \\ &= 2 \int_0^{\infty} \left(\int_0^Q \int_x^Q f_S(y + \mu_S) dy dx + \int_0^Q \int_Q^{\infty} f_S(y + \mu_S) dy dx \right) f_i(\mu_i - Q) dQ \end{aligned}$$

$$\begin{aligned}
&= 2 \int_0^\infty \left(\int_0^Q \int_x^\infty f_S(y + \mu_S) dy dx \right) f_i(\mu_i - Q) dQ \\
&= \int_0^\infty \left(\int_0^Q (2 - 2F_S(x + \mu_S)) dx \right) f_i(\mu_i - Q) dQ \\
&= \int_0^\infty \left(2Q - 2 \int_0^Q (F_S(x + \mu_S)) dx \right) f_i(\mu_i - Q) dQ.
\end{aligned}$$

The first equality holds by Lemma 1 and $q^T = \mu_T$ for all $T \subseteq N$. The second equality holds since $\mu_{S \cup \{i\}} = \mu_S + \mu_i$. The fourth equality holds since $((\mu_S + \delta) - \mu_S)f_S(\mu_S + \delta)$ is equal to $(\mu_S - (\mu_S - \delta))f_S(\mu_S - \delta)$, and $(\mu_i - (\mu_i - \delta)) * (1 - F_S(\mu_S - (\mu_i - \delta) + \mu_i))$ is equal to $((\mu_i + \delta) - \mu_i) * F_S(\mu_S - ((\mu_i + \delta) - \mu_i))$ for all $\delta \in \mathbb{R}^+$, and f_S and f_i are symmetric functions. The sixth equality follows from changing variables $Q = \mu_i - x$ and $y = z - \mu_S$.

Let $i \in N$ and $S \subset T \subseteq N/\{i\}$. Then

$$\begin{aligned}
v(T \cup \{i\}) &- v(T) - (v(S \cup \{i\}) - v(S)) = G(T, i) - G(S, i) \\
&= \int_0^\infty \left(2Q - 2 \int_0^Q (F_T(y + \mu_T)) dy \right) f_i(\mu_i - Q) dQ \\
&\quad - \int_0^\infty \left(2Q - 2 \int_0^Q (F_S(y + \mu_S)) dy \right) f_i(\mu_i - Q) dQ \\
&= 2 \int_0^\infty \left(\int_0^Q (F_S(y + \mu_S) - F_T(y + \mu_T)) dy \right) f_i(\mu_i - Q) dQ.
\end{aligned}$$

So, to prove convexity, it suffices to show that $F_S(x) - F_T(x + \mu_{T/S}) \geq 0$ for all $x \geq \mu_S$.

$$\begin{aligned}
F_S(x) &- F_T(x + \mu_{T/S}) = \int_{-\infty}^\infty f_{T/S}(y) dy F_S(x) - \int_{-\infty}^\infty f_{T/S}(y) F_S(x + \mu_{T/S} - y) dy \\
&= 2 * \int_{\mu_{T/S}}^\infty f_{T/S}(y) F_S(x) dy - \int_{\mu_{T/S}}^\infty f_{T/S}(y) F_S(x + \mu_{T/S} - y) dy \\
&\quad - \int_{-\infty}^{\mu_{T/S}} f_{T/S}(z) F_S(x + \mu_{T/S} - z) dz \\
&= 2 * \int_{\mu_{T/S}}^\infty f_{T/S}(y) F_S(x) dy - \int_{\mu_{T/S}}^\infty f_{T/S}(y) F_S(x + \mu_{T/S} - y) dy \\
&\quad - \int_{\mu_{T/S}}^\infty f_{T/S}(2\mu_{T/S} - y) F_S(x - \mu_{T/S} + y) dy \\
&= 2 * \int_{\mu_{T/S}}^\infty f_{T/S}(y) F_S(x) dy - \int_{\mu_{T/S}}^\infty f_{T/S}(y) F_S(x + \mu_{T/S} - y) dy
\end{aligned}$$

$$\begin{aligned}
& - \int_{\mu_{T/S}}^{\infty} f_{T/S}(y) F_S(x - \mu_{T/S} + y) dy \\
& = \int_{\mu_{T/S}}^{\infty} f_{T/S}(y) \left(2 * F_S(x) - F_S(x + \mu_{T/S} - y) - F_S(x - \mu_{T/S} + y) \right) dy \quad (8)
\end{aligned}$$

The first equality holds by $\int_{-\infty}^{\infty} f_{T/S}(y) dy = 1$ and F_T being the convolution of $F_{T \setminus S}$ and F_S . The third equality follows from changing variable $z = 2\mu_{T/S} - y$. The fourth equality follows by symmetry of $f_{T/S}$ around $\mu_{T/S}$. Let $D(x, y)$ be the term between parentheses in the last equality. So,

$$D(x, y) = 2 * F_S(x) - F_S(x + \mu_{T/S} - y) - F_S(x - \mu_{T/S} + y).$$

Then,

$$\begin{aligned}
D(x, \mu_{T/S}) &= 0 \text{ for all } x, \\
\frac{\partial D(x, y)}{\partial y} &= f_S(x + \mu_{T/S} - y) - f_S(x - \mu_{T/S} + y) \geq 0 \text{ for all } x \geq \mu_S \text{ and } y \geq \mu_{T/S}.
\end{aligned}$$

The last inequality holds since f_S is unimodal and symmetric around the mean, and $x + \mu_{T/S} - y$ is closer to μ_S than $x - \mu_{T/S} + y$ if $x \geq \mu_S$ and $y \geq \mu_{T/S}$.

So, we conclude that $2 * F_S(x) - F_S(x + \mu_{T/S} - y) - F_S(x - \mu_{T/S} + y)$ is nonnegative for all $x \geq \mu_S$ and $y \geq \mu_{T/S}$. Since $f_{T/S}$ is nonnegative, (8) is nonnegative for all $x \geq \mu_S$, which completes the proof. □

This result is restrictive since it proves convexity only for our special case with $1 - c/p = 1/2$ (i.e., $p = 2c$). However, for most of the situations (e.g., newsvendor situations with independent symmetric strictly unimodal demand distributions), the convexity is strict at optimal fractile $1/2$. Hence, for the situations with $1 - c/p \approx 1/2$ (i.e., $p \approx 2c$), the associated games are also convex. Besides, $p = 2c$ implies 100 percent profit margin.

The following proposition shows another property of games with independent symmetric demand distributions.

Proposition 1 *Consider two newsvendor situations $\Gamma = (N, (X_i)_{i \in N}, c, p)$ with $1 - \frac{c}{p} < \frac{1}{2}$ and $\bar{\Gamma} = (N, (X_i)_{i \in N}, \bar{c}, p)$ with $\bar{c} = p - c$, where both situations have independent and symmetric demands. Then (N, v^Γ) is convex if and only if $(N, v^{\bar{\Gamma}})$ is convex.*

Proof: To prove the proposition we will show that the difference between marginal contributions of a retailer to a bigger and a smaller coalition is the same for both situations. Consider coalition $T \subseteq N$. Using the symmetry of demand, we can write that

$$\begin{aligned}
v^\Gamma(T) &= p \int_0^{1-\frac{c}{p}} F_T^{-1}(z) dz \\
&= p \int_0^{\frac{c}{p}} F_T^{-1}(z) dz - p \int_{1-\frac{c}{p}}^{\frac{c}{p}} F_T^{-1}(z) dz \\
&= p \int_0^{\frac{c}{p}} F_T^{-1}(z) dz - (2c - p)\mu_T \\
&= p \int_0^{1-\frac{\bar{c}}{p}} F_T^{-1}(z) dz - (p - 2\bar{c})\mu_T \\
&= v^{\bar{\Gamma}}(T) - (p - 2\bar{c})\mu_T.
\end{aligned} \tag{9}$$

The first equality follows from (5). The third equality holds since symmetry of demand distributions implies that $\int_0^a F_T^{-1}(1/2 + x) dx + \int_0^a F_T^{-1}(1/2 - x) dx = 2a * \mu_T$. The fourth equality follows from changing $c = p - \bar{c}$. The last equality follows from (5) again.

Let $i \in N$ and $S \subset T \subseteq N/\{i\}$, then from (9), it follows that

$$\begin{aligned}
v^\Gamma(T \cup \{i\}) &= v^\Gamma(T) - v^\Gamma(S \cup \{i\}) + v^\Gamma(S) \\
&= v^{\bar{\Gamma}}(T \cup \{i\}) - v^{\bar{\Gamma}}(T) - v^{\bar{\Gamma}}(S \cup \{i\}) + v^{\bar{\Gamma}}(S) \\
&\quad + (p - 2\bar{c})(\mu_{T \cup \{i\}} - \mu_T - \mu_{S \cup \{i\}} + \mu_S) \\
&= v^{\bar{\Gamma}}(T \cup \{i\}) - v^{\bar{\Gamma}}(T) - v^{\bar{\Gamma}}(S \cup \{i\}) + v^{\bar{\Gamma}}(S).
\end{aligned}$$

The last equality holds since $\mu_{T \cup \{i\}} - \mu_T = \mu_i = \mu_{S \cup \{i\}} - \mu_S$. This completes the proof. \square

4 Normal demand distributions

In this section, we study newsvendor situations with independent normal demand distributions and show that their associated games are convex regardless of the optimal fractile.⁵

⁵The result presented in this section is taken, with permission of the authors, from the unpublished working paper Slikker et al. (2001).

Since we would like to show convexity for all possible $1 - c/p$, we transfer our analysis to a higher dimension. Consider a group of retailers N facing random demand $(X_i)_{i \in N}$ with a selling price p . Using the general characteristic function (6), convexity conditions in (1) can be transformed as follows:

$$\begin{aligned} v_{T \cup \{i\}}(y) - v_T(y) &\geq v_{S \cup \{i\}}(y) - v_S(y) \\ \text{for all } i \in N, \text{ all } S, T \subseteq N \setminus \{i\} \text{ with } S \subset T \text{ and all } y \in [0, 1]. \end{aligned} \quad (10)$$

Note that since newsvendor games are superadditive, convexity conditions with $S = \emptyset$ are already satisfied.

Let $i \in N$ and $S, T \subseteq N \setminus \{i\}$ such that $S \subset T$. The so-called convexity function associated with the convexity condition for i, S, T is defined by

$$\begin{aligned} \text{con}_{i,S,T}(y) &= v_{T \cup \{i\}}(y) - v_T(y) - v_{S \cup \{i\}}(y) + v_S(y) \\ &= p \int_0^y F_{T \cup \{i\}}^{-1}(z) dz - p \int_0^y F_T^{-1}(z) dz - p \int_0^y F_{S \cup \{i\}}^{-1}(z) dz + p \int_0^y F_S^{-1}(z) dz \\ &= p \left(\int_0^y F_{T \cup \{i\}}^{-1}(z) dz - \int_0^y F_T^{-1}(z) dz - \int_0^y F_{S \cup \{i\}}^{-1}(z) dz + \int_0^y F_S^{-1}(z) dz \right) \end{aligned} \quad (11)$$

Assume that demands are independent with $X_i \sim \text{Norm}(\mu_i, \sigma_i)$ for all $i \in N$. Then demand $X_S = \sum_{i \in S} X_i$ of a coalition $S \subseteq N$ is also normally distributed with parameters $\mu_S = \sum_{i \in S} \mu_i$ and $\sigma_S = \sqrt{\sum_{i \in S} (\sigma_i)^2}$. The following lemma derives a relation between standard deviations of demands of several coalitions.

Lemma 2 *Let N be a finite set and let $X_i \sim \text{Norm}(\mu_i, \sigma_i)$ for all $i \in N$ be independent stochastic demands. For all $i \in N$ and $S, T \subseteq N \setminus \{i\}$ with $S \subset T$ it holds that*

$$\sigma_{S \cup \{i\}} - \sigma_S \geq \sigma_{T \cup \{i\}} - \sigma_T.$$

Proof: Let $i \in N$ and $S, T \subseteq N \setminus \{i\}$ with $S \subset T$. Then

$$\begin{aligned} \sigma_{S \cup \{i\}} - \sigma_S &= \sqrt{\sum_{j \in S \cup \{i\}} (\sigma_j)^2} - \sqrt{\sum_{j \in S} (\sigma_j)^2} = \frac{(\sigma_i)^2}{\sqrt{\sum_{j \in S \cup \{i\}} (\sigma_j)^2} + \sqrt{\sum_{j \in S} (\sigma_j)^2}} \\ &\geq \frac{(\sigma_i)^2}{\sqrt{\sum_{j \in T \cup \{i\}} (\sigma_j)^2} + \sqrt{\sum_{j \in T} (\sigma_j)^2}} = \sigma_{T \cup \{i\}} - \sigma_T, \end{aligned}$$

where the last equality follows similar to the first two.

□

Let $D_{\mu,\sigma}$ be the distribution function of normal distribution $Norm(\mu, \sigma)$. So, we can rewrite the convexity function as follows:

$$\begin{aligned} con_{i,S,T}(y) = p & \left(\int_0^y D_{\mu_{T \cup \{i\}}, \sigma_{T \cup \{i\}}}^{-1}(z) dz - \int_0^y D_{\mu_T, \sigma_T}^{-1}(z) dz \right. \\ & \left. - \int_0^y D_{\mu_{S \cup \{i\}}, \sigma_{S \cup \{i\}}}^{-1}(z) dz + \int_0^y D_{\mu_S, \sigma_S}^{-1}(z) dz \right). \end{aligned}$$

The following theorem states that newsvendor games with independent normal demand distributions are convex.

Theorem 2 *Let $(N, (X_i)_{i \in N}, c, p)$ be a newsvendor situation with independent stochastic demands $X_i \sim Norm(\mu_i, \sigma_i)$ for all $i \in N$. The associated newsvendor game (N, v) is convex.*

Proof: In the proof, we show that convexity requirements (10) are satisfied. In other words, we show that $con_{i,S,T}(y) \geq 0$ for all $i \in N$, all $S, T \subseteq N \setminus \{i\}$ with $S \subset T$ and all $y \in [0, 1]$. Let $i \in N$ and $S, T \subseteq N \setminus \{i\}$ with $S \subset T$. Then, by Proposition 1, it is sufficient to show $con_{i,S,T}(y) \geq 0$ for all $y \in [0, 1/2]$ only. Without loss of generality, we assume that $p = 1$.

Let $con_{i,S,T}^{(1)}(y)$ be the first derivative of $con_{i,S,T}(y)$, which is given by

$$con_{i,S,T}^{(1)}(y) = D_{\mu_{T \cup \{i\}}, \sigma_{T \cup \{i\}}}^{-1}(y) - D_{\mu_T, \sigma_T}^{-1}(y) - D_{\mu_{S \cup \{i\}}, \sigma_{S \cup \{i\}}}^{-1}(y) + D_{\mu_S, \sigma_S}^{-1}(y).$$

Consider a normal distribution with $D_{\mu,\sigma}$, then $D_{\mu,\sigma}^{-1}(y) = \mu + k_y \sigma$, where k_α is the unique real number such that $P(X < k_\alpha) = \alpha$ with X having standard normal distribution $Norm(0, 1)$. Therefore, we can rewrite $con_{i,S,T}^{(1)}(y)$ as follows:

$$\begin{aligned} con_{i,S,T}^{(1)}(y) &= \mu_{T \cup \{i\}} + k_y \sigma_{T \cup \{i\}} - \mu_T - k_y \sigma_T - \mu_{S \cup \{i\}} - k_y \sigma_{S \cup \{i\}} + \mu_S + k_y \sigma_S \\ &= k_y (\sigma_{T \cup \{i\}} - \sigma_T - \sigma_{S \cup \{i\}} + \sigma_S) \\ &\geq 0 \quad \text{for all } y \in [0, 1/2]. \end{aligned} \tag{12}$$

The second equality follows by $\mu_{T \cup \{i\}} - \mu_T = \mu_{\{i\}} = \mu_{S \cup \{i\}} - \mu_S$. The inequality holds by Lemma 2 and $k_\alpha \leq 0$ for all $\alpha \in [0, 1/2]$. Since $con_{i,S,T}(0) = 0$ and from (12), it follows that $con_{i,S,T}(y) \geq 0$ for all $y \in [0, 1/2]$. This completes the proof. \square

5 Uniform demand distributions

Convexity states that the marginal contribution of a player to a coalition increases with the size of the coalition. In this section, we investigate this property of convex games in the class of newsvendor games with independent uniform demand distributions. First, in section 5.1, we analyze the marginal contribution of a retailer with uniform demand distribution to small coalitions by focusing on a special class of 3-person games with identical, independent uniform demand distributions. Then, in section 5.2, we investigate the marginal contribution of a retailer with uniform demand distribution to large coalitions.

5.1 3-person games

In this subsection, we analyse 3-person newsvendor situations with identical, independent uniform demand distributions and show that the associated newsvendor games are convex regardless of the value of $1 - c/p$.

Consider a group of 3 retailers with identical independent uniform demand distributions $U(a, b)$ with $b > a \geq 0$. Without loss of generality, we fix $a = 0$. Since all retailers are identical, the demand distribution and the value of coalition S depends on the cardinality of S only. Let F_k be the demand distribution function of k -person coalitions. Furthermore, let us introduce the general characteristic function of k -person coalitions $v_k(y)$, which is given by

$$v_k(y) = p \int_0^y F_k^{-1}(z) dz. \quad (13)$$

The convexity conditions in (10) can be reduced to the following condition for this 3-retailer situation:

$$v_3(y) - v_2(y) \geq v_2(y) - v_1(y) \quad \text{for all } y \in [0, 1].$$

Related with this convexity condition, we define the following convexity function on $[0, 1]$:

$$\begin{aligned} \text{con}(y) &= v_3(y) - 2 * v_2(y) + v_1(y) \\ &= p \int_0^y F_3^{-1}(z) dz - 2p \int_0^y F_2^{-1}(z) dz + p \int_0^y F_1^{-1}(z) dz \\ &= p \left(\int_0^y F_3^{-1}(z) dz - 2 \int_0^y F_2^{-1}(z) dz + \int_0^y F_1^{-1}(z) dz \right). \end{aligned}$$

The second equality follows by (13).

We remark that for convexity, by Proposition 1, it is sufficient to show that $\text{con}(y) \geq 0$ for all possible $y \in [0, 1/2]$.

Since the demands are $U(0, b)$, using convolution formula (7), we can derive that

$$\begin{aligned} F_1(x) &= \begin{cases} 0 & \text{if } x \leq 0; \\ \frac{x}{b} & \text{if } 0 < x \leq b; \\ 1 & \text{if } b < x; \end{cases} \\ F_2(x) &= \begin{cases} 0 & \text{if } x \leq 0; \\ \frac{x^2}{2b^2} & \text{if } 0 < x \leq b; \\ \frac{x^2}{2b^2} - 2\frac{(x-b)^2}{2b^2} & \text{if } b < x \leq 2b; \\ 1 & \text{if } 2b < x; \end{cases} \\ F_3(x) &= \begin{cases} 0 & \text{if } x \leq 0; \\ \frac{x^3}{6b^3} & \text{if } 0 < x \leq b; \\ \frac{x^3}{6b^3} - \frac{3(x-b)^3}{6b^3} & \text{if } b < x \leq 2b; \\ \frac{x^3}{6b^3} - \frac{3(x-b)^3}{6b^3} + \frac{3(x-2b)^3}{6b^3} & \text{if } 2b < x \leq 3b; \\ 1 & \text{if } 3b < x. \end{cases} \end{aligned}$$

Furthermore,

$$\begin{aligned} F_1^{-1}(y) &= by \quad \text{if } 0 < y < 1; \\ F_2^{-1}(y) &= \begin{cases} \sqrt{2b^2y} & \text{if } 0 < y \leq \frac{1}{2}; \\ 2b - \sqrt{2b^2(1-y)} & \text{if } \frac{1}{2} < y < 1. \end{cases} \end{aligned}$$

Note that $F_1^{-1}(0)$ and $F_2^{-1}(0)$ are set-valued. However, we assume for technical reasons that $F_1^{-1}(0) = F_2^{-1}(0) = 0$.

We can write the derivative of the convexity function $\text{con}(y)$ as follows:

$$\text{con}^{(1)}(y) = p \left(F_3^{-1}(y) - 2F_2^{-1}(y) + F_1^{-1}(y) \right).$$

Without loss of generality, we fix $p = 1$ in the remainder of the document unless stated explicitly. In the following part of this section, we deal with situations such that $0 \leq y = 1 - c/p \leq 1/6$ and situations such that $1/6 \leq y = 1 - c/p \leq 1/2$, separately. The following lemma shows that newsvendor games with 3 retailers having identical independent uniform demand distributions are convex if $0 \leq y = 1 - c/p \leq 1/6$.

Lemma 3 *Let $(N, (X_i)_{i \in N}, c, p)$ be a newsvendor situation with $N = \{1, 2, 3\}$, $X_i \sim U(0, b)$ for all $i \in N$ and $c \geq \frac{5}{6}p$. The associated newsvendor game is convex.*

Proof: To prove the theorem, we show that

$$\text{con}(y) \geq 0 \quad \text{for } 0 \leq y \leq \frac{1}{6}.$$

Recall the derivative of the convexity function

$$\text{con}^{(1)}(y) = F_3^{-1}(y) - 2F_2^{-1}(y) + F_1^{-1}(y).$$

Let us define $H(x) = \text{con}^{(1)}(F_3(x))$. Note that since $F_3(x) = \frac{x^3}{6b^3}$ is strictly increasing on the interval $(0, b)$, it has a strictly increasing inverse function. Therefore, if $H(x)$ is positive on the interval $(0, b)$ then so is $\text{con}^{(1)}(y)$ on the interval $(0, 1/6)$. Moreover, $H(0) = \text{con}^{(1)}(0)$ and $H(b) = \text{con}^{(1)}(1/6)$.

Consider $0 \leq x \leq b$, then

$$\begin{aligned} H(x) &= F_3^{-1}\left(\frac{x^3}{6b^3}\right) - 2F_2^{-1}\left(\frac{x^3}{6b^3}\right) + F_1^{-1}\left(\frac{x^3}{6b^3}\right) \\ &= x - 2\sqrt{2b^2 \frac{x^3}{6b^3}} + b \frac{x^3}{6b^3} \\ &= x - 2\sqrt{\frac{x^3}{3b}} + \frac{x^3}{6b^2}. \end{aligned}$$

Taking further derivatives, we derive that

$$\begin{aligned} H^{(1)}(x) &= 1 - \sqrt{\frac{3x}{b}} + \frac{x^2}{2b^2} \quad \text{for } 0 \leq x \leq b; \\ H^{(2)}(x) &= -\sqrt{\frac{3}{b}} \frac{1}{2} \frac{1}{\sqrt{x}} + \frac{x}{b^2} \quad \text{for } 0 < x \leq b; \\ H^{(3)}(x) &= \sqrt{\frac{3}{b}} \frac{1}{4} \frac{1}{\sqrt{x^3}} + \frac{1}{b^2} > 0 \quad \text{for } 0 < x \leq b. \end{aligned} \tag{14}$$

Then we can further calculate the following values:

$$\begin{aligned} H(0) &= 0; \\ H(b) &= b - 2\sqrt{\frac{b^3}{3b}} + \frac{b^3}{6b^2} \\ &= b - 2\frac{b}{\sqrt{3}} + \frac{b}{6} \\ &= \left(\frac{7}{6} - \frac{2}{\sqrt{3}}\right)b > 0; \\ H^{(1)}(0) &= 1; \end{aligned}$$

$$\begin{aligned}
H^{(1)}(b) &= 1 - \sqrt{3} + \frac{1}{2} < 0; \\
\lim_{x \rightarrow 0} H^{(2)}(x) &\rightarrow -\infty; \\
H^{(2)}(b) &= -\frac{\sqrt{3}}{2b} + \frac{1}{b} > 0.
\end{aligned}$$

A summary and conclusions can be found in Table 1.

Table 1: Derivatives of convexity function

	$x = 0$	$0 < x < b$	$x = b$
$H^{(3)}(x)$		+	+
$H^{(2)}(x)$		$- \Rightarrow +$	+
$H^{(1)}(x)$	+	$+ \Rightarrow -$	-
$H(x)$	0	+	+
	$y = 0$	$0 < y < 1/6$	$y = 1/6$
$con^{(1)}(y)$	0	+	+

In the table the second and last columns except the last row are known from the calculations above. We fill the remaining column as follows. Cell (2,3) is positive from equation (14). Cell (3,3) is first negative and then positive since $\lim_{x \rightarrow 0} H^{(2)}(x) < 0$, $H^{(2)}(b) > 0$ and $H^{(3)}(x) > 0$ for all $x \in (0, b)$. For Cell (4,3) we have the following explanation: $H^{(1)}(0) > 0$ and $H^{(1)}(b) < 0$. Since $H^{(2)}(x)$ is negative first, $H^{(1)}(x)$ is going down and becomes negative (starting positive) and remains negative even after $H^{(2)}(x)$ becomes positive. For Cell (5,3) we have a similar explanation: $H(0) = 0$ and $H(b) > 0$. Since $H^{(1)}(x)$ is positive first, $H(x)$ starts to increase (starting from zero) and stays positive even after $H^{(1)}(x)$ becomes negative. From the fifth row, the last row follows immediately.

Since $con(0) = 0$ and $con^{(1)}(y) \geq 0$ for all $y \in [0, 1/6]$, $con(y) \geq 0$ for all $y \in [0, 1/6]$. This completes the proof.

□

We use the following technical lemma to show that newsvendor games with 3 retailers having identical independent uniform demand distributions are convex if $1/6 \leq y = 1 - c/p \leq 1/2$.

Lemma 4 *Let $b > 0$, then*

$$\frac{27b[x^2 - 3(x-b)^2]^2}{2[3b(x^3 - 3(x-b)^3)]^{\frac{3}{2}}} \geq \frac{3\sqrt{3}}{2b} - \frac{3\sqrt{3}}{b^2}(x-b) \quad \text{for } b \leq x \leq \frac{3b}{2}.$$

The proof of this lemma is given in the appendix.

The following lemma shows that newsvendor games with 3 retailers, and identical, independent, uniform demand distributions are convex if $1/6 \leq y = 1 - c/p \leq 1/2$.

Lemma 5 *Let $(N, (X_i)_{i \in N}, c, p)$ be a newsvendor situation with $N = \{1, 2, 3\}$, $X_i \sim U(0, b)$ for all $i \in N$ and $\frac{1}{2}p \leq c \leq \frac{5}{6}p$. The associated newsvendor game is convex.*

Proof: To prove the theorem, we show that

$$\text{con}(y) \geq 0 \quad \text{for } \frac{1}{6} \leq y \leq \frac{1}{2}.$$

Recall the first derivative of the convexity function

$$\text{con}^{(1)}(y) = F_3^{-1}(y) - 2F_2^{-1}(y) + F_1^{-1}(y). \quad (15)$$

As in the proof of Lemma 3, let $H(x) = \text{con}^{(1)}(F_3(x))$. Note that since $F_3(x) = \frac{x^3}{6b^3} - 3\frac{(x-b)^3}{6b^3}$ is strictly increasing on the interval $(b, 3b/2)$, it has a strictly increasing inverse function. Therefore if $H(x)$ changes sign only once (to negative) on the interval $(b, 3b/2)$ so does $\text{con}^{(1)}(y)$ on the interval $(1/6, 1/2)$. Moreover, $H(b) = \text{con}^{(1)}(1/6)$ and $H(3b/2) = \text{con}^{(1)}(1/2)$.

Consider $b \leq x \leq 3b/2$, then

$$\begin{aligned} H(x) &= F_3^{-1}\left(\frac{x^3}{6b^3} - 3\frac{(x-b)^3}{6b^3}\right) - 2F_2^{-1}\left(\frac{x^3}{6b^3} - 3\frac{(x-b)^3}{6b^3}\right) + F_1^{-1}\left(\frac{x^3}{6b^3} - 3\frac{(x-b)^3}{6b^3}\right) \\ &= x - 2\sqrt{2b^2\left(\frac{x^3}{6b^3} - 3\frac{(x-b)^3}{6b^3}\right)} + b\left(\frac{x^3}{6b^3} - 3\frac{(x-b)^3}{6b^3}\right). \end{aligned}$$

Taking further derivatives, we derive that for $b \leq x \leq 3b/2$

$$\begin{aligned} H^{(1)}(x) &= 1 + (x^2 - 3(x-b)^2) \left[\frac{1}{2b^2} - \frac{3}{\sqrt{(x^3 - 3(x-b)^3)3b}} \right]; \\ H^{(2)}(x) &= [2x - 6(x-b)] \left[\frac{1}{2b^2} - \frac{3}{\sqrt{(x^3 - 3(x-b)^3)3b}} \right] \\ &\quad + \frac{27b[x^2 - 3(x-b)^2]^2}{2[3b(x^3 - 3(x-b)^3)]^{\frac{3}{2}}}. \end{aligned} \quad (16)$$

Considering the first term of $H^{(2)}(x)$, we note that

$$\begin{aligned}
& [2x - 6(x-b)] \left[\frac{1}{2b^2} - \frac{3}{\sqrt{(x^3 - 3(x-b)^3)3b}} \right] \\
& \geq [2x - 6(x-b)] \left[\frac{1}{2b^2} - \frac{3}{\sqrt{3}b^2} \right] \\
& = \frac{4\sqrt{3}-2}{b^2}(x-b) + \frac{1-2\sqrt{3}}{b} \quad \text{for } b \leq x \leq \frac{3b}{2},
\end{aligned} \tag{17}$$

where the inequality holds since $\sqrt{(x^3 - 3(x-b)^3)3b} \geq \sqrt{3}b^2$ and $2x - 6(x-b) \geq 0$ for $b \leq x \leq 3b/2$.

From Lemma 4, we know that

$$\frac{27b[x^2 - 3(x-b)^2]^2}{2[3b(x^3 - 3(x-b)^3)]^{\frac{3}{2}}} \geq \frac{3\sqrt{3}}{2b} - \frac{3\sqrt{3}}{b^2}(x-b) \quad \text{for } b \leq x \leq \frac{3b}{2}. \tag{18}$$

From expression (16), (17) and (18), we infer that

$$\begin{aligned}
H^{(2)}(x) & \geq \frac{4\sqrt{3}-2}{b^2}(x-b) + \frac{1-2\sqrt{3}}{b} + \frac{3\sqrt{3}}{2b} - \frac{3\sqrt{3}}{b^2}(x-b) \\
& = \frac{2-\sqrt{3}}{2b} + \frac{\sqrt{3}-2}{b^2}(x-b) > 0 \quad \text{for } b < x < \frac{3b}{2}.
\end{aligned} \tag{19}$$

Furthermore, we calculate the following values:

$$\begin{aligned}
H(b) & = \left(\frac{7}{6} - \frac{2}{3}\sqrt{3}\right)b > 0; \\
H\left(\frac{3b}{2}\right) & = 0; \\
H^{(1)}(b) & = \frac{3}{2} - \sqrt{3} < 0; \\
H^{(1)}\left(\frac{3b}{2}\right) & = \frac{1}{4}.
\end{aligned}$$

We summarize and conclude with Table 2.

In Table 2, the second and last columns except the last row are known from the calculations above. We fill the remaining columns as follows. Cell (2, 3) is positive from equation (19). Cell (3, 3) is first negative and then positive since $H^{(1)}(b) < 0$, $H^{(1)}(3b/2) > 0$ and $H^{(2)}(x) > 0$ for all $x \in (b, 3b/2)$. For cell (4, 3) we have the following explanation: $H(b) > 0$ and $H(3b/2) = 0$. Since $H^{(1)}(x)$ is negative first, $H(x)$ is going down and becomes negative (starting positive) and stays negative even after $H^{(1)}(x)$ becomes positive. From the fourth row, the last row follows immediately.

Table 2: Derivatives of convexity function

	$x = b$	$b < x < 3b/2$	$x = 3b/2$
$H^{(2)}(x)$		+	
$H^{(1)}(x)$	−	$− \Rightarrow +$	+
$H(x)$	+	$+ \Rightarrow −$	0
	$y = 1/6$	$1/6 < y < 1/2$	$y = 1/2$
$con^{(1)}(y)$	+	$+ \Rightarrow −$	0

From Theorem 1 and the proof of Lemma 3, we know that

$$con(\frac{1}{2}) \geq 0; \quad (20)$$

$$con(\frac{1}{6}) \geq 0. \quad (21)$$

From (20), (21) and since $con^{(1)}$ (starting from positive) changes sign only once (to negative) in the interval $(1/6, 1/2)$, we conclude that $con(y) \geq 0$ for $1/6 \leq y \leq 1/2$. This completes the proof.

□

From Lemmas 3 and 5, and Proposition 1 the following theorem follows immediately.

Theorem 3 *Let $(N, (X_i)_{i \in N}, c, p)$ be a newsvendor situation, where $N = \{1, 2, 3\}$ and the retailers face identical independent uniformly distributed demands. The associated game is convex.*

Since the problem gets more complex, it is not possible to apply the technic of analysis above to investigate situations with nonidentical uniform demands. However, we conjecture that newsvendor games with three retailers all having independent uniform demand distributions are still convex.

5.2 Marginal contributions to large coalitions

In this subsection, we focus on marginal contributions of a retailer to large coalitions, where all retailers have independent uniform demand distributions. The central limit theorem states that the distribution of the sum of n independent random variables approaches a normal distribution for sufficiently large n . Hence, for large coalitions, normal

distributions provide us with good approximations for their demand distributions. We consider two large coalitions and assume that they have normal demand distributions. We show that the contribution of the retailer with uniform demand to the larger coalition is higher than his contribution to the smaller coalition. Since the bigger coalition consists of the smaller one and several additional retailers, the bigger coalition is assumed to have higher mean and standard deviation.

So, consider two coalitions T and S such that $S \subset T$. Suppose that $X_S \sim \text{Norm}(\mu_S, \sigma_S)$ and $X_T \sim \text{Norm}(\mu_T, \sigma_T)$ with $\mu_T \geq \mu_S$, $\sigma_T \geq \sigma_S$. Furthermore, consider a retailer i with uniform demand distribution $U(0, b)$ with $b > 0$ joining these two coalitions. The difference in the contribution of retailer i to coalition T and S for optimal fractile $y \in [0, 1]$ is given by the following convexity function:

$$\begin{aligned} \text{con}_{i,S,T}(y) &= v_{T \cup \{i\}}(y) - v_T(y) - v_{S \cup \{i\}}(y) + v_S(y) \\ &= \int_0^y F_{T \cup \{i\}}^{-1}(z) dz - \int_0^y F_T^{-1}(z) dz - \int_0^y F_{S \cup \{i\}}^{-1}(z) dz + \int_0^y F_S^{-1}(z) dz. \end{aligned}$$

The following theorem shows that the marginal contribution of retailer i to coalition T is bigger than its contribution to coalition S .

Theorem 4 *Let $(N, (X_j)_{j \in N}, c, p)$ be a newsvendor situation and let (N, v) be the associated newsvendor game. Let T , S and i be two coalitions and a retailer such that $i \notin T \subseteq N$ and $S \subset T$. Assume that coalition T and S have normal demand distributions $\text{Norm}(\mu_T, \sigma_T)$ and $\text{Norm}(\mu_S, \sigma_S)$ such that $\mu_T \geq \mu_S$, $\sigma_T \geq \sigma_S$, respectively. Let retailer i have uniform demand $U(0, b)$. Then,*

$$v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S).$$

Proof: To prove the theorem, we need to show that $\text{con}_{i,S,T}(y) \geq 0$ for all $y \in [0, 1]$. Proposition 1 and Theorem 1 imply that it is sufficient to show that $\text{con}_{i,S,T}(y) \geq 0$ for $y = [0, 1/2)$. Since $\text{con}_{i,S,T}(0) = 0$, it is sufficient to show that for all $y \in [0, 1/2)$

$$0 \leq \text{con}_{i,S,T}^{(1)}(y) = F_{T \cup \{i\}}^{-1}(y) - F_T^{-1}(y) - F_{S \cup \{i\}}^{-1}(y) + F_S^{-1}(y).$$

Since we could not derive the inverse functions explicitly, we transfer the problem to the following one.

Consider a $y \in [0, 1/2)$. Let $k < 0$ be such that $\Theta(k) = y$, where Θ is the standard normal distribution function. Choose $\Delta_1 \in \mathbb{R}$ such that $F_{T \cup \{i\}}(\mu_T + (k + \Delta_1)\sigma_T) = y$.

In other words,

$$\begin{aligned} F_T^{-1}(y) &= D_{\mu_T, \sigma_T}^{-1}(y) = \mu_T + k\sigma_T; \\ F_S^{-1}(y) &= D_{\mu_S, \sigma_S}^{-1}(y) = \mu_S + k\sigma_S; \\ F_{T \cup \{i\}}^{-1}(y) &= \mu_T + (k + \Delta_1)\sigma_T. \end{aligned}$$

Claim 1: Δ_1 is unique and satisfies $0 < \Delta_1 < b'/2$, where $b' = b/\sigma_T$.

The proof of this claim is given in the appendix.

Using Claim 1, we derive

$$\begin{aligned} \text{con}_{i,S,T}^{(1)}(y) &= F_{T \cup \{i\}}^{-1}(y) - F_T^{-1}(y) - F_{S \cup \{i\}}^{-1}(y) + F_S^{-1}(y) \\ &= \mu_T + (k + \Delta_1)\sigma_T - \mu_T - k\sigma_T - F_{S \cup \{i\}}^{-1}(y) + \mu_S + k\sigma_S \\ &= \Delta_1\sigma_T - F_{S \cup \{i\}}^{-1}(y) + \mu_S + k\sigma_S \\ &= \mu_S + (k + \Delta_1 \frac{\sigma_T}{\sigma_S})\sigma_S - F_{S \cup \{i\}}^{-1}(y). \end{aligned}$$

Therefore, showing that $F_{S \cup \{i\}}(\mu_S + (k + \Delta_1 \frac{\sigma_T}{\sigma_S})\sigma_S) \geq y$ is sufficient to prove that $\text{con}_{i,S,T}^{(1)}(y) \geq 0$.

$$\begin{aligned} F_{S \cup \{i\}}(\mu_S + (k + \Delta_1 \frac{\sigma_T}{\sigma_S})\sigma_S) &= \int_{-\infty}^{\infty} f_{\{i\}}(z) F_S(\mu_S + (k + \Delta_1 \frac{\sigma_T}{\sigma_S})\sigma_S - z) dz \\ &= \frac{1}{b} \int_0^b D_{\mu_S, \sigma_S}(\mu_S + (k + \Delta_1 \frac{\sigma_T}{\sigma_S})\sigma_S - z) dz \\ &= \frac{\sigma_S}{b} \int_0^{\frac{b}{\sigma_S}} \Theta(k + \Delta_1 \frac{\sigma_T}{\sigma_S} - u) du \\ &= \frac{1}{\bar{a}b'} \int_0^{\bar{a}b'} \Theta(k + \bar{a}\Delta_1 - u) du \end{aligned}$$

where $b' = b/\sigma_T$ and $\bar{a} = \sigma_T/\sigma_S$. The first equality follows from convolution formula (7). The second equality holds since $X_i \sim U(0, b)$ and $X_S \sim \text{Norm}(\mu_S, \sigma_S)$. The third equality follows from the change in coordinates $z = u\sigma_S$ and the fact that $D_{\mu, \sigma}(x) = \Theta((x - \mu)/\sigma)$. Since $\sigma_S \leq \sigma_T$, we have $\bar{a} \geq 1$. Moreover, we have

$$y = F_{T \cup \{i\}}(\mu_T + (k + \Delta_1)\sigma_T) = \frac{1}{b'} \int_0^{b'} \Theta(k + \Delta_1 - u) du, \quad (22)$$

where the first equality follows from the definition of Δ_1 and second equality holds by similar argument as above.

In the following part, we show that $F_{S \cup \{i\}}(\mu_S + (k + \Delta_1 \frac{\sigma_T}{\sigma_S})\sigma_S) \geq y$.

We define the function $m : [0, \infty) \rightarrow [0, \infty)$ by

$$m(a) = \frac{1}{b'} \int_0^{ab'} \Theta(k + a\Delta_1 - u) du.$$

So we can write

$$\begin{aligned} F_{S \cup \{i\}}(\mu_S + (k + \Delta_1 \frac{\sigma_T}{\sigma_S})\sigma_S) &= \frac{1}{\bar{a}b'} \int_0^{\bar{a}b'} \Theta(k + \bar{a}\Delta_1 - u) du \\ &= \frac{m(\bar{a})}{\bar{a}}, \end{aligned}$$

where $\bar{a} \geq 1$.

We derive the further derivatives of the function m as follows:

$$\begin{aligned} m^{(1)}(a) &= \frac{1}{b'} \left[b' \Theta(k - a(b' - \Delta_1)) + \int_0^{ab'} \Delta_1 \theta(k + a\Delta_1 - u) du \right] \\ &= \frac{1}{b'} \left[b' \Theta(k - a(b' - \Delta_1)) - \Delta_1 (\Theta(k - a(b' - \Delta_1)) - \Theta(k + a\Delta_1)) \right] \\ &= \frac{1}{b'} \left[\Delta_1 \Theta(k + a\Delta_1) + (b' - \Delta_1) \Theta(k - a(b' - \Delta_1)) \right]; \\ m^{(2)}(a) &= \frac{1}{b'} \left[\Delta_1^2 \theta(k + a\Delta_1) - (b' - \Delta_1)^2 \theta(k - a(b' - \Delta_1)) \right]; \\ m^{(3)}(a) &= \frac{1}{b'} \left[-\Delta_1^3 \theta(k + a\Delta_1) \right. \\ &\quad \left. - (b' - \Delta_1)^3 \theta(k - a(b' - \Delta_1)) \right]. \end{aligned} \tag{23}$$

The last expression follows from $\theta^{(1)}(x) = -x\theta(x)$.

The function m has the following properties:

$$\begin{aligned} m(0) &= 0; \\ m(1) &= \frac{1}{b'} \int_0^{b'} \Theta(k + \Delta_1 - u) du = y \quad (\text{see (22)}); \\ m^{(1)}(a) &= \frac{1}{b'} \left[\Delta_1 \Theta(k + a\Delta_1) + (b' - \Delta_1) \Theta(k - a(b' - \Delta_1)) \right] > 0 \quad \text{for every } a \in (0, \infty); \\ m^{(1)}(0) &= \frac{1}{b'} \left[\Delta_1 \Theta(k) + (b' - \Delta_1) \Theta(k) \right] = \frac{b'}{b'} \Theta(k) = y. \end{aligned}$$

Claim 2 Consider the function m . There is exactly one $a^* \in (0, \infty)$ such that $m^{(2)}(a^*) = 0$. Furthermore $m^{(2)}(a) < 0$ for all $a \in [0, a^*)$ and $m^{(2)}(a) > 0$ for all $a \in (a^*, \infty)$.

The proof of this claim is given in the appendix.

From Lemma 6, which can be found in the appendix as well, it follows that $\frac{m(a)}{a} \geq m(1) = y$ for all $a \in [1, \infty)$.

Therefore

$$F_{S \cup \{i\}}(\mu_S + (k + \Delta_1 \frac{\sigma_T}{\sigma_S})\sigma_S) = \frac{m(\bar{a})}{\bar{a}} \geq y$$

The last equality follows from $\bar{a} \geq 1$. This completes the proof.

□

In this section, we focused on a group of retailers with independent uniform demand distributions. We investigated two cases, where a retailer joins to small coalitions and big coalitions. Since the normal distribution provides a good approximation for the demand distribution of very big coalitions by the central limit theorem, we assumed normal demand distributions for these coalitions. For both of these cases, we were able to show that the marginal contribution of the retailer increases with the size of the coalition as convexity suggests. However, we still do not know whether this relation holds for medium-sized coalitions. We conjecture anyhow that newsvendor games with independent uniform demand distributions are convex.

6 A counterexample

The results in the previous sections point towards a general result. One might conjecture that any newsvendor situation with independent symmetric unimodal demand distributions results in a cooperative game that is convex. In this section, however, we present an example that contradicts this conjecture.

Example 6.1 Consider newsvendor situation $\Gamma = (N, (X_i)_{i \in N}, c, p)$, such that $N = \{1, 2, 3\}$, $c = 49$, $p = 50$, and independent stochastic demands $X_1 = X_2 \sim U(0, 2)$ and X_3 with density function

$$f_{\{3\}}(x) = \begin{cases} 0 & \text{if } x < 0 ; \\ 1/50 & \text{if } 0 \leq x < 2; \\ 230/25 & \text{if } 2 \leq x < 2.1; \\ 1/50 & \text{if } 2.1 \leq x < 4.1; \\ 0 & \text{if } 4.1 \leq x. \end{cases}$$

Using (4), we write the value of any coalition S as follows:

$$v^\Gamma(S) = p(q^S y - \int_0^{q^S} F_S(u) du),$$

where $y = 1 - c/p$ is the optimal fractile and q^S is the optimal order quantity.

In this example, we focus on the contribution of retailer 1 to coalitions $\{3\}$ and $\{2, 3\}$.

Note that

$$\begin{aligned} F_{\{3\}}(x) &= \int_0^x f_{\{3\}}(z) dz = \frac{x}{50} \quad \text{if } 0 < x \leq 2, \\ F_{\{1,3\}}(x) &= \int_0^2 f_{\{1\}}(z) F_{\{3\}}(x-z) dz = \frac{1}{2} \int_0^2 F_{\{3\}}(x-z) dz = \frac{x^2}{200} \quad \text{if } 0 < x \leq 2, \\ F_{\{2,3\}}(x) &= \int_0^2 f_{\{2\}}(z) F_{\{3\}}(x-z) dz = \frac{1}{2} \int_0^2 F_{\{3\}}(x-z) dz = \frac{x^2}{200} \quad \text{if } 0 < x \leq 2, \\ F_{\{1,2,3\}}(x) &= \int_0^2 f_{\{2\}}(z) F_{\{1,3\}}(x-z) dz = \frac{1}{2} \int_0^2 F_{\{1,3\}}(x-z) dz = \frac{x^3}{1200} \quad \text{if } 0 < x \leq 2. \end{aligned}$$

Moreover, $F_{\{1,2\}}(x)$ is given by the convolution of two identical uniform distributions, which can be found in section 4.

The optimal fractile is calculated as $y = 1 - c/p = 1/50$. Recall that the optimal order quantity of coalition S is $q^S = F_S^{-1}(y)$. So, the optimal order quantities of coalitions $\{3\}$, $\{1, 3\}$ and $\{2, 3\}$ are calculated as $q^{\{3\}} = 1$, $q^{\{1,3\}} = q^{\{2,3\}} = 2$. From $F_{\{1,2,3\}}(x) = \int_0^{4.1} f_{\{3\}}(z) F_{\{1,2\}}(x-z) dz$, it can be calculated that $F_{\{1,2,3\}}(2.4) \approx 0.02557 > 1/50$. Moreover, $F_{\{1,2,3\}}(2) = 1/150 < 1/50$. Hence, we conclude that $2 < q^{\{1,2,3\}} < 2.4$.

The contribution of retailer 1 to coalition $\{3\}$ is given by

$$\begin{aligned} v^\Gamma(\{1, 3\}) - v^\Gamma(\{3\}) &= p \left(q^{\{1,3\}} y - \int_0^{q^{\{1,3\}}} F_{\{1,3\}}(u) du - q^{\{3\}} y + \int_0^{q^{\{3\}}} F_{\{3\}}(u) du \right) \\ &= p \left(2y - \int_0^2 F_{\{1,3\}}(u) du - y + \int_0^1 F_{\{3\}}(u) du \right) \\ &= p \left(\frac{2}{50} - \int_0^2 \frac{u^2}{200} du - \frac{1}{50} + \int_0^1 \frac{u}{50} du \right) \\ &= p \frac{100}{6000} = p \times \text{Size}(A), \end{aligned}$$

where $\text{Size}(A)$ corresponds to the size of area in Figure 1.

The contribution of retailer 1 to coalition $\{2, 3\}$ is given by

$$v^\Gamma(\{1, 2, 3\}) - v^\Gamma(\{2, 3\})$$

$$\begin{aligned}
&= p \left(q^{\{1,2,3\}} y - \int_0^{q^{\{1,2,3\}}} F_{\{1,2,3\}}(u) du - q^{\{2,3\}} y + \int_0^{q^{\{2,3\}}} F_{\{2,3\}}(u) du \right) \\
&= p \left((q^{\{1,2,3\}} - 2)y - \int_2^{q^{\{1,2,3\}}} F_{\{1,2,3\}}(u) du \right. \\
&\quad \left. + \int_0^2 (F_{\{2,3\}}(u) - F_{\{1,2,3\}}(u)) du \right) \\
&< p(\text{Size}(C) + \text{Size}(B)),
\end{aligned}$$

where $\text{Size}(B)$ and $\text{Size}(C)$ correspond to the size of the areas in Figure 1. The last inequality holds since $(q^{\{1,2,3\}} - 2)y - \int_2^{q^{\{1,2,3\}}} F_{\{1,2,3\}}(u) du < \text{Size}(C)$ and $\text{Size}(B) = \int_0^2 (F_{\{2,3\}}(u) - F_{\{1,2,3\}}(u)) du$. $\text{Size}(B)$ and $\text{Size}(C)$ can be calculated as follows.

$$\begin{aligned}
\text{Size}(B) &= \int_0^2 (F_{\{2,3\}}(u) - F_{\{1,2,3\}}(u)) du \\
&= \int_0^2 \left(\frac{u^2}{200} - \frac{u^3}{1200} \right) du = \frac{60}{6000}; \\
\text{Size}(C) &= \left(\frac{1}{50} - \frac{1}{150} \right) (q^{\{1,2,3\}} - 2) < \left(\frac{1}{50} - \frac{1}{150} \right) (2.4 - 2) = \frac{32}{6000}.
\end{aligned}$$

The last inequality holds since $q^{\{1,2,3\}} < 2.4$.

Hence, we derive that

$$v^\Gamma(\{1, 3\}) - v^\Gamma(\{3\}) = p \frac{100}{6000} > p \frac{92}{6000} > p(\text{Size}(C) + \text{Size}(B)) > v^\Gamma(\{1, 2, 3\}) - v^\Gamma(\{2, 3\}).$$

Therefore, we conclude that (N, v^Γ) is not convex.

◇

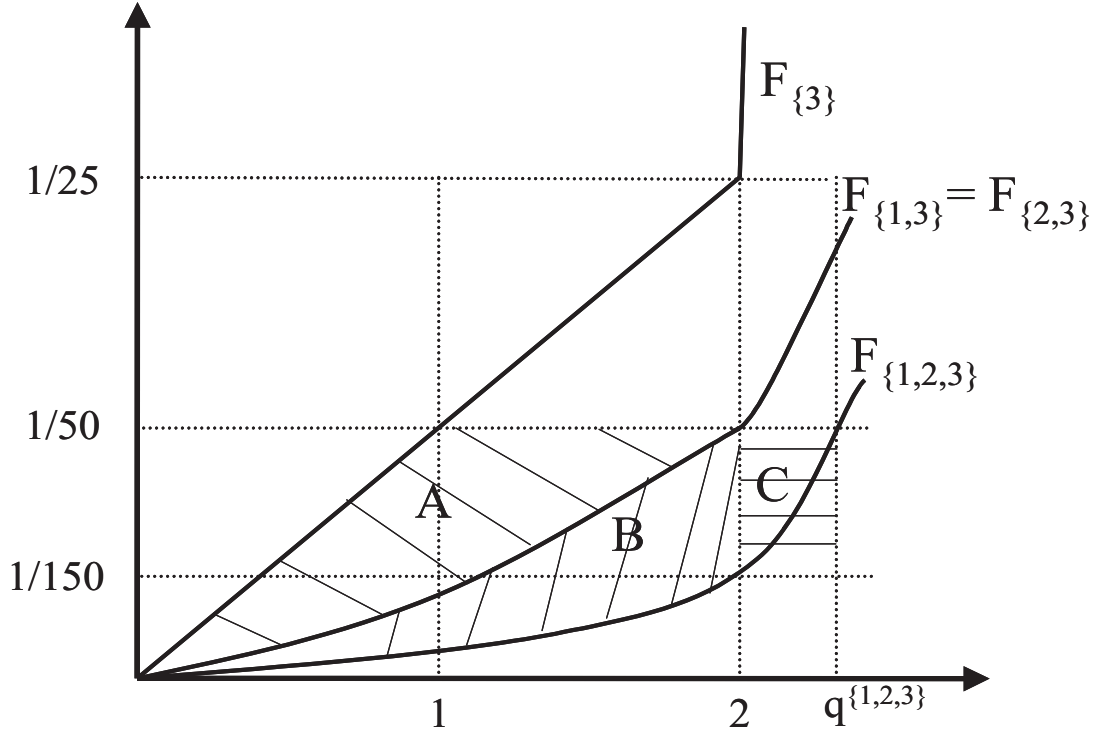


Figure 1: Contribution of retailer 1

7 Concluding Remarks

The demand distribution of the retailers and the optimal fractile, which is determined by the ratio of the purchasing cost to selling price, are two important factors that affect the convexity of newsvendor games. In this paper, we concentrated on a class of newsvendor games with independent symmetric unimodal demand distributions and we identified several subclasses to be convex. Finally, we provided a counterexample showing that convexity does not hold for all games in this class. Despite this, we think that there might still be unidentified subclasses of convex games which are not covered in this study. There are several reasons for us to think so. First of all, in this study, we showed that all newsvendor games with $p = 2c$ in the class of newsvendor games with independent symmetric unimodal distributions are convex. Furthermore, we showed that, for the newsvendor games with normal distributions, convexity is a quite natural property regardless of the optimal fractile. Independent symmetric unimodal distributions are known to be well shaped and their convolution tends to approach normal distribution quickly. Therefore, we expect the property of increasing marginal contributions as convexity suggests to hold for big coalitions formed by retailers with independent symmetric

unimodal distributions. Secondly, uniform distributions are the distributions with the lowest kurtosis, namely 1.8 (a measure of normality, see De Carlo (1997) for a discussion on kurtosis) among symmetric unimodal distributions, where the kurtosis for normal distributions is 3. Hence, uniform distributions are on one side furthest away from normal distributions within this class (we remark that in Example 6.1, the kurtosis of demand distribution of retailer 3 is larger than 21). Despite this, we were able to show the property of increasing marginal contributions for retailers with uniform demand distributions, when they join small and large coalitions. Furthermore, we conjecture that newsvendor games with independent uniform demand distributions are convex.

Appendix

This appendix contains the proofs of Lemma 1, Lemma 4 and two claims in the proof of Theorem 4. Furthermore, we present Lemma 6 here.

Proof of Lemma 1: Consider the value of $S \cup T$:

$$\begin{aligned}
v(S \cup T) &= -c * q^{S \cup T} + p * E_{X_{S \cup T}}[\min(q^{S \cup T}, X_{S \cup T})] \\
&= -c * q^{S \cup T} + p * E_{X_{S \cup T}}[\min(q^{S \cup T}, X_{S \cup T})] + p * E_{X_{S \cup T}}[\min(q^S + q^T, X_{S \cup T})] \\
&\quad - p * E_{X_{S \cup T}}[\min(q^S + q^T, X_{S \cup T})] \\
&= -c * q^{S \cup T} + p * E_{X_{S \cup T}}[\min(q^S + q^T, X_{S \cup T})] \\
&\quad + p * E_{X_{S \cup T}}[(X_{S \cup T} - (q^S + q^T))^+ - (X_{S \cup T} - q^{S \cup T})^+].
\end{aligned} \tag{24}$$

The first equality follows from (3). The second equality follows by adding and subtracting the same term. The last equality follows from $\min(q^{S \cup T}, X_{S \cup T}) = X_{S \cup T} - (X_{S \cup T} - q^{S \cup T})^+$ and $\min(q^S + q^T, X_{S \cup T}) = X_{S \cup T} - (X_{S \cup T} - (q^S + q^T))^+$. In the final expression, the first term is the purchasing cost of ordering $q^{S \cup T}$ units, the second term is the expected revenue if the coalition orders $q^S + q^T$ units and the final term is the extra expected revenue of ordering $q^{S \cup T}$ instead of $q^S + q^T$.

The sales of coalition $S \cup T$ with order quantity $q^S + q^T$ can be written as follows:

$$\begin{aligned}
\min(q^S + q^T, X_{S \cup T}) &= \min(q^S, X_S) + \min(q^T, X_T) \\
&\quad + \min(q^S + q^T - \min(q^S, X_S) - \min(q^T, X_T), X_{S \cup T}),
\end{aligned}$$

$$\begin{aligned}
& X_S + X_T - \min(q^S, X_S) - \min(q^T, X_T) \\
= & \min(q^S, X_S) + \min(q^T, X_T) \\
& + \min((q^S - X_S)^+, (X_T - q^T)^+) \\
& + \min((q^T - X_T)^+, (X_S - q^S)^+).
\end{aligned}$$

After the first equality, the first term is the sales in the local markets using local orders and the second term is the sales if there is excess demand in one local market and excess stock in the other. The last equality follows from the fact that if $X_S \geq q^S$ and $X_T \geq q^T$ ($X_S \leq q^S$ and $X_T \leq q^T$), $q^S + q^T - \min(q^S, X_S) - \min(q^T, X_T) = 0$ ($X_S + X_T - \min(q^S, X_S) - \min(q^T, X_T) = 0$), and if $X_S \leq q^S$ and $X_T \geq q^T$ ($X_S \geq q^S$ and $X_T \leq q^T$), $q^S + q^T - \min(q^S, X_S) - \min(q^T, X_T) = q^S - X_S$ and $X_S + X_T - \min(q^S, X_S) - \min(q^T, X_T) = X_T - q^T$ ($q^S + q^T - \min(q^S, X_S) - \min(q^T, X_T) = q^T - X_T$ and $X_S + X_T - \min(q^S, X_S) - \min(q^T, X_T) = X_S - q^S$).

Consequently, the expected sales of coalition $S \cup T$ with order quantity $q^S + q^T$ is given by

$$\begin{aligned}
E_{X_{S \cup T}}[\min(q^S + q^T, X_{S \cup T})] &= E_{X_S}[\min(q^S, X_S)] + E_{X_T}[\min(q^T, X_T)] \\
&+ E_{X_{S \cup T}}[\min((q^S - X_S)^+, (X_T - q^T)^+)] \\
&+ E_{X_{S \cup T}}[\min((q^T - X_T)^+, (X_S - q^S)^+)]. \quad (25)
\end{aligned}$$

Hence the expected sales of coalition $S \cup T$ with order quantity $q^S + q^T$ is the expected sales of coalition S with order quantity q^S and coalition T with order quantity q^T , and the expected sales coming from exchange of excess goods to satisfy excess demand.

Using this, we derive

$$\begin{aligned}
G(S, T) &= v(S \cup T) - v(S) - v(T) \\
&= -c * q^{S \cup T} + p * E_{X_{S \cup T}}[\min(q^S + q^T, X_{S \cup T})] \\
&\quad + p * E_{X_{S \cup T}}[(X_{S \cup T} - (q^S + q^T))^+ - (X_{S \cup T} - q^{S \cup T})^+] \\
&\quad + c * q^S - p * E_{X_S}[\min(q^S, X_S)] + c * q^T - p * E_{X_T}[\min(q^T, X_T)] \\
&= -c * (q^{S \cup T} - q^S - q^T) + p * E_{X_{S \cup T}}[(X_{S \cup T} - (q^S + q^T))^+ - (X_{S \cup T} - q^{S \cup T})^+] \\
&\quad + p(E_{X_{S \cup T}}[\min(q^S + q^T, X_{S \cup T})] - E_{X_S}[\min(q^S, X_S)] - E_{X_T}[\min(q^T, X_T)]) \\
&= -c * (q^{S \cup T} - q^S - q^T) + p * E_{X_{S \cup T}}[(X_{S \cup T} - (q^S + q^T))^+ - (X_{S \cup T} - q^{S \cup T})^+] \\
&\quad + p * E_{X_{S \cup T}}[\min((q^S - X_S)^+, (X_T - q^T)^+)] \\
&\quad + p * E_{X_{S \cup T}}[\min((q^T - X_T)^+, (X_S - q^S)^+)].
\end{aligned}$$

The first equality holds by definition. The second equality holds by (24) and (3). The third equality follows from rewriting. The last equality holds by (25).

□

Proof of Lemma 4: Let $1 \leq z \leq \frac{3}{2}$. We first show that

$$\frac{27}{2} \frac{(z^2 - 3(z-1)^2)^2}{3\sqrt{3}(\frac{1}{2} - (z-1))} \geq (3(z^3 - 3(z-1)^3))^{\frac{3}{2}}$$

by proving that $(z^2 - 3(z-1)^2)^4 - (3-2z)^2(z^3 - 3(z-1)^3)^3 \geq 0$, since

$$\begin{aligned} & \frac{27}{2} \frac{(z^2 - 3(z-1)^2)^2}{3\sqrt{3}(\frac{1}{2} - (z-1))} \geq (3(z^3 - 3(z-1)^3))^{\frac{3}{2}} \\ \iff & \left(\frac{27}{2} \frac{(z^2 - 3(z-1)^2)^2}{3\sqrt{3}(\frac{1}{2} - (z-1))} \right)^2 \geq (3(z^3 - 3(z-1)^3))^3 \\ \iff & \frac{(z^2 - 3(z-1)^2)^4}{(3-2z)^2} \geq (z^3 - 3(z-1)^3)^3 \\ \iff & (z^2 - 3(z-1)^2)^4 - (3-2z)^2(z^3 - 3(z-1)^3)^3 \geq 0. \end{aligned}$$

The second expression follows from taking the square of both sides and since $z^3 - 3(z-1)^3 \geq 0$ for $1 \leq z \leq \frac{3}{2}$.

Then,

$$\begin{aligned} & (z^2 - 3(z-1)^2)^4 - (3-2z)^2(z^3 - 3(z-1)^3)^3 \\ &= 32z^{11} - 528z^{10} + 3744z^9 - 15032z^8 + 37986z^7 \\ & \quad - 63597z^6 + 72225z^5 - 55836z^4 + 28917z^3 - 9612z^2 + 1863z - 162 \\ &= 3(z-1) + 12(z-1)^2 + 17(z-1)^3 + 28(z-1)^4 + 29(z-1)^5 \\ & \quad - 191(z-1)^6 - 286(z-1)^7 + 184(z-1)^8 + 224(z-1)^9 - 176(z-1)^{10} + 32(z-1)^{11} \\ &= (z-1)^7 \left[\frac{3}{(z-1)^6} + \frac{12}{(z-1)^5} - 286 \right] + (z-1)^6 \left[\frac{17}{(z-1)^3} + \frac{28}{(z-1)^2} + \frac{29}{(z-1)} - 191 \right] \\ & \quad + 184(z-1)^8 + (z-1)^{10} \left[\frac{224}{(z-1)} - 176 \right] + 32(z-1)^{11} \\ &\geq 290(z-1)^7 + 115(z-1)^6 + 184(z-1)^8 + 272(z-1)^{10} + 32(z-1)^{11} \geq 0. \end{aligned}$$

The first equality follows from expanding the expression. The second equality follows from rewriting using the Taylor expansion of a polynomial, e.g., $d(a+h) = d(a) + d^{(1)}(a)h + 1/2!d^{(2)}(a)h^2 + \dots$. The first inequality follows from replacing $(z-1)$'s in the denominators with $\max_{z \in (1, 3/2)}(z-1) = 1/2$. So we conclude that

$$\frac{27}{2} \frac{(z^2 - 3(z-1)^2)^2}{3\sqrt{3}(\frac{1}{2} - (z-1))} \geq (3(z^3 - 3(z-1)^3))^{\frac{3}{2}} \quad \text{for } 1 \leq z \leq \frac{3}{2}.$$

Applying the variable change $z = x/b$, we obtain that for $b \leq x \leq \frac{3b}{2}$,

$$\frac{27}{2} \frac{((\frac{x}{b})^2 - 3(\frac{x}{b} - 1)^2)^2}{3\sqrt{3}(\frac{1}{2} - (\frac{x}{b} - 1))} \geq (3((\frac{x}{b})^3 - 3(\frac{x}{b} - 1)^3))^{\frac{3}{2}}$$

$$\begin{aligned}
&\Longleftrightarrow \frac{27b^2}{2b^6} \frac{((x)^2 - 3(x-b)^2)^2}{3\sqrt{3}(\frac{1}{2} - (\frac{x-b}{b}))} \geq \frac{(3b((x)^3 - 3(x-b)^3))^{\frac{3}{2}}}{b^6} \\
&\Longleftrightarrow \frac{27b^2}{2} \frac{((x)^2 - 3(x-b)^2)^2}{3\sqrt{3}(\frac{1}{2} - (\frac{x-b}{b}))} \geq (3b((x)^3 - 3(x-b)^3))^{\frac{3}{2}} \\
&\Longleftrightarrow \frac{27b[x^2 - 3(x-b)^2]^2}{2[3b(x^3 - 3(x-b)^3)]^{\frac{3}{2}}} \geq \frac{3\sqrt{3}}{2b} - \frac{3\sqrt{3}}{b^2}(x-b).
\end{aligned}$$

This completes the proof. □

Proof of Claim 1: Since $F_{T \cup \{i\}}$ is a strictly increasing function, it is sufficient to show that $F_{T \cup \{i\}}(\mu_T + k\sigma_T) < y$ and $F_{T \cup \{i\}}(\mu_T + (k + \frac{1}{2}b')\sigma_T) > y$ to prove the claim. Since $F_{T \cup \{i\}}(x) < F_T(x)$ for all $x \in \mathbb{R}$, $F_{T \cup \{i\}}(\mu_T + k\sigma_T) < y$. So it remains to show $F_{T \cup \{i\}}(\mu_T + (k + \frac{1}{2}b')\sigma_T) > y$, where

$$\begin{aligned}
F_{T \cup \{i\}}(\mu_T + (k + \frac{1}{2}b')\sigma_T) &= \frac{1}{b} \int_0^b D_{\mu_T, \sigma_T}(\mu_T + (k + \frac{1}{2}b')\sigma_T - z) dz \\
&= \frac{\int_0^{\frac{1}{2}b} (D_{\mu_T, \sigma_T}(\mu_T + k\sigma_T + u) + D_{\mu_T, \sigma_T}(\mu_T + k\sigma_T - u)) du}{b}
\end{aligned}$$

The first equality follows from the convolution of $Norm(\mu_T, \sigma_T)$ and $U(0, b)$. The second equality follows from $b' = b/\sigma_T$ and rewriting the expression.

Furthermore,

$$\begin{aligned}
&d_{\mu_T, \sigma_T}(\mu_T + k\sigma_T + a) > d_{\mu_T, \sigma_T}(\mu_T + k\sigma_T - a) && \text{for all } a \geq 0 \\
\implies &D_{\mu_T, \sigma_T}(\mu_T + k\sigma_T + a) - y > y - D_{\mu_T, \sigma_T}(\mu_T + k\sigma_T - a) && \text{for all } a \geq 0 \\
\implies &D_{\mu_T, \sigma_T}(\mu_T + k\sigma_T + a) + D_{\mu_T, \sigma_T}(\mu_T + k\sigma_T - a) > 2y && \text{for all } a \geq 0 \\
\implies &\int_0^{\frac{1}{2}b} (D_{\mu_T, \sigma_T}(\mu_T + k\sigma_T + u) + D_{\mu_T, \sigma_T}(\mu_T + k\sigma_T - u)) du > yb && \text{for all } a \geq 0 \\
\implies &F_{T \cup \{i\}}(\mu_T + (k + \frac{1}{2}b')\sigma_T) > y && \text{for all } a \geq 0.
\end{aligned}$$

The first expression holds by $k < 0$ and $a \geq 0$. The second expression holds by $D_{\mu_T, \sigma_T}(\mu_T + k\sigma_T) = y$. This completes the proof. □

Proof of Claim 2: In the proof, we show that $m^{(2)}$ has exactly one positive root, $m^{(2)}(0) < 0$ and $m^{(3)}(a^*) > 0$.

$$\begin{aligned}
m^{(2)}(a) &= \frac{1}{b'} \left[\Delta_1^2 \theta(k + a\Delta_1) - (b' - \Delta_1)^2 \theta(k - a(b' - \Delta_1)) \right] = 0 \\
\iff \Delta_1^2 \theta(k + a\Delta_1) - (b' - \Delta_1)^2 \theta(k - a(b' - \Delta_1)) &= 0 \\
\iff \Delta_1^2 \theta(k + a\Delta_1) &= (b' - \Delta_1)^2 \theta(k - a(b' - \Delta_1)) \\
\iff \Delta_1^2 \frac{e^{-(k+a\Delta_1)^2/2}}{\sqrt{2\pi}} &= (b' - \Delta_1)^2 \frac{e^{-(k-a(b' - \Delta_1))^2/2}}{\sqrt{2\pi}} \\
\iff e^{(k-a(b' - \Delta_1))^2/2 - (k+a\Delta_1)^2/2} &= \frac{(b' - \Delta_1)^2}{\Delta_1^2} \\
\iff (k - a(b' - \Delta_1))^2 - (k + a\Delta_1)^2 &= 2 \ln \left(\frac{(b' - \Delta_1)^2}{\Delta_1^2} \right) \\
\iff ((b' - \Delta_1)^2 - \Delta_1^2)a^2 - 2kb'a - 2 \ln \left(\frac{(b' - \Delta_1)^2}{\Delta_1^2} \right) &= 0.
\end{aligned} \tag{26}$$

so the two roots of $m^{(2)}$ are

$$\begin{aligned}
a^* &= \frac{2kb' + \sqrt{(-2kb')^2 + 8((b' - \Delta_1)^2 - \Delta_1^2) \ln \left(\frac{(b' - \Delta_1)^2}{\Delta_1^2} \right)}}{2((b' - \Delta_1)^2 - \Delta_1^2)} > 0; \\
a' &= \frac{2kb' - \sqrt{(-2kb')^2 + 8((b' - \Delta_1)^2 - \Delta_1^2) \ln \left(\frac{(b' - \Delta_1)^2}{\Delta_1^2} \right)}}{2((b' - \Delta_1)^2 - \Delta_1^2)} < 0.
\end{aligned}$$

The inequalities hold since $k < 0$ and $\Delta_1 < b'/2$ from Claim 1, which results in

$$8((b' - \Delta_1)^2 - \Delta_1^2) \ln \left(\frac{(b' - \Delta_1)^2}{\Delta_1^2} \right) > 0.$$

Furthermore,

$$m^{(2)}(0) = \frac{1}{b'} \left[\Delta_1^2 \theta(k) - (b' - \Delta_1)^2 \theta(k) \right] < 0$$

since $b' - \Delta_1 > \Delta_1$.

Since $b' - \Delta_1 > \Delta_1 > 0$, $k < 0$ and $a^* > 0$,

$$|k + a^* \Delta_1| < |k - a^*(b' - \Delta_1)|$$

and

$$k - a^*(b' - \Delta_1) < 0.$$

Hence,

$$\Delta_1(k + a^* \Delta_1) < -(b' - \Delta_1)(k - a^*(b' - \Delta_1)). \tag{27}$$

Since a^* is root of $m^{(2)}$, from (26),

$$\Delta_1^2 \theta(k + a^* \Delta_1) = (b' - \Delta_1)^2 \theta(k - a^*(b' - \Delta_1)) > 0. \tag{28}$$

The inequality follows since $\Delta_1 > 0$ and $\theta(x) \geq 0$ for all $x \in \mathbb{R}$.

From (27) and (28), it follows that

$$\begin{aligned} \Delta_1^3(k + a^*\Delta_1)\theta(k + a^*\Delta_1) &< -(b' - \Delta_1)^3(k - a^*(b' - \Delta_1))\theta(k - a^*(b' - \Delta_1)) \\ \implies -\Delta_1^3(k + a^*\Delta_1)\theta(k + a^*\Delta_1) - (b' - \Delta_1)^3(k - a^*(b' - \Delta_1))\theta(k - a^*(b' - \Delta_1)) &> 0. \end{aligned}$$

Therefore, from (23), we have

$$\begin{aligned} m^{(3)}(a^*) = \frac{1}{b'} &\left[-\Delta_1^3(k + a^*\Delta_1)\theta(k + a^*\Delta_1) \right. \\ &\left. - (b' - \Delta_1)^3(k - a^*(b' - \Delta_1))\theta(k - a^*(b' - \Delta_1)) \right] > 0. \end{aligned}$$

This completes the proof. □

The following lemma is used in the proof of Theorem 4.

Lemma 6 *Let $C : [0, \infty) \rightarrow [0, \infty)$ be a 2 times differentiable function, such that*

- $C(0) = 0$,
- *There is a $t^* \in [0, \infty)$ such that $C^{(2)}(t^*) = 0$, $C^{(2)}(t) < 0$ for all $t \in [0, t^*)$, and $C^{(2)}(t) > 0$ for all $t \in (t^*, \infty)$,*
- $C(1) = C^{(1)}(0)$.

Then $\frac{C(t)}{t} \geq C(1)$ for all $t \in [1, \infty)$.

Proof: To prove the theorem, we will first show that there is a $t' \in (0, 1)$ such that $C^{(1)}(t') = C^{(1)}(0)$ and $t^* \in (0, t')$.

We know that C is continuous on $[0, 1]$, differentiable on $(0, 1)$. So from the mean value theorem, there exist a $t' \in (0, 1)$ such that

$$C^{(1)}(t') = \frac{C(1) - C(0)}{1 - 0} = C(1).$$

Similarly, $C^{(1)}$ is continuous on $[0, t']$ and differentiable on $(0, t')$. Furthermore, $C^{(1)}(t') = C(1) = C^{(1)}(0)$. From Rolle's theorem, there exist a $\hat{t} \in (0, t')$ such that $C^{(2)}(\hat{t}) = 0$. By assumption it follows that $\hat{t} = t^*$.

Knowing that $C^{(2)}$ starts negative, becomes 0 at t^* and continues positive, it is obvious that $C^{(2)}(t) > 0$ for all $t \in (t', \infty)$. Hence, $C^{(1)}(t) > C^{(1)}(t') = C(1)$ for all $t \in [1, \infty)$.

Therefore,

$$\frac{C(t)}{t} = \frac{C(1) + \int_1^t C^{(1)}(x)dx}{t} \geq \frac{C(1) + (t-1)C(1)}{t} = C(1) \quad \text{for all } t \in [1, \infty).$$

This completes the proof.

□

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